THE GROTHENDIECK GROUP

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Let us start by looking at the natural numbers $\mathbb{N}$. Adding two natural numbers gives a natural number again. Add 0 to this set, and we gain an "identity" element, that is, a number which when added to any natural number n gives n itself. But we would like more. What number when added to a natural number n gives the identity? Can we "subtract" one natural number from another? In order to do this, we need to move from the semigroup structure of $\mathbb{N}$ to a group.

Let $(R, +)$ be an abelian semigroup with identity 0. A semigroup is a set R along with a binary operation $+ : R \times R \to R$ that satisfies the associative property, i.e. $a + (b + c) = (a + b) + c$. Some easy examples of semigroups are $(\mathbb{N}, +)$ and $(\mathbb{Q}, \cdot)$.

If we have an abelian semigroup with an identity element, that is, an element $0 \in R$ which satisfies $a + 0 = a\, \forall a \in R$, then we would like to ask the following question: for $a \in R$, does there exist an element $b \in R$ such that $a + b = 0$? That is, we look for "inverses" of the elements of R. In other words, can we subtract elements of R? One can, by passing onto what is known as the Grothendieck or universal group of R.

Let $X = R \times R$. Define an equivalence relation on X as follows: $(a, b) \sim (c, d)$ iff $\exists\, e \in R$ such that $a + d + e = b + c + e$. Why not simply define two elements (a, b) and (c, d) to be equivalent if $a + d = b + c$? This will become clear shortly.

We must prove that $\sim$ is indeed an equivalence relation. Reflexivity and symmetry can be seen clearly by the commutativity of R. Let us prove transitivity. Suppose $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$. Then $\exists\, e_1, e_2 \in R$ such that

$$a + d + e_1 = b + c + e_1 \tag{1}$$

and

$$c + f + e_2 = d + e + e_2. \tag{2}$$

Adding (1) and (2) gives

$$a + f + (c + d + e_1 + e_2) = b + e + (c + d + e_1 + e_2). \tag{3}$$

Hence $(a, b) \sim (e, f)$. It should now be clear why the equivalence relation has been defined in this way, for cancellation need not hold in the semigroup.

Now, let $G(R) = X/\sim$. We will show that $G(R)$ is a group under the binary operation

$$[(a, b)] \oplus [(c, d)] = [(a + c, b + d)].$$

First, we must show that this operation is well-defined. Suppose $[(a, b)] = [(a', b')]$ and $[(c, d)] = [(c', d')]$. Then $\exists\, e_1, e_2 \in R$ such that

$$a + b' + e_1 = a' + b + e_1 \tag{4}$$

and

$$c + d' + e_2 = c' + d + e_2. \tag{5}$$

Adding (4) and (5) shows that the operation $\oplus$ is indeed well-defined.

The identity element of $G(R)$ is $[(0,0)]$. But a quick verification shows that $[(0,0)] = [(a,a)]\,\forall a \in R$. This immediately indicates that the inverse of an element $[(a,b)]$ should be $[(b,a)]$, i.e., $[(b,a)] =$"$\ominus[(a,b)]$".

We would like for R to sit inside $G(R)$ in some way. Let $a \in R$. We send this to the element $[(a,0)]$ in $G(R)$ and denote it simply by $[a]$. Then it is clear that $G(R)$ is exactly equal to the set $\{[a] \ominus [b] : a, b \in R\}$. That is, $G(R)$ is a set in which the formal differences of elements of R are legalised. Let the map $a \mapsto [a]$ be denoted by $i : R \to G(R)$. It is an easy observation that the map i is injective iff cancellation holds in R, that is, if $a + c = b + c$, then $a = b$. It is surjective iff R is already a group.

Let us go back to the question we started with, of how to find the inverses of natural numbers under addition. Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The Grothendieck group $G(\mathbb{N}_0)$ is then exactly equal to the set of integers, $\mathbb{Z}$.

We now look at the semigroup $R = \mathbb{N}_0 \cup \{\infty\}$, where ∞ is an element of R satisfying the following property:

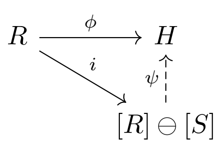
$$a + \infty = \infty \quad \forall a \in R.$$

What is $G(R)$ in this case? The answer is unexpected. $G(R)$ turns out to be the trivial group $\{0\}$! Let us consider arbitrary $[a] \in G(R)$. Then $[a] \oplus [\infty] = [a + \infty] = [\infty]$. But since $G(R)$ is a group, one can cancel terms on both sides to get $[a] = 0$. What if we consider $\mathbb{N}$ with multiplication as the operation? Then $G(R) = \mathbb{Q}^+$, the set of positive rational numbers. How then do we arrive at the set of all rational numbers $\mathbb{Q}$?

Suppose we had started with an abelian semigroup R and a sub-semigroup S. Consider the set $R \times S$ and a relation defined as follows: $(r,s) \sim (r',s')$ iff $\exists\, s'' \in S$ such that $r + s' + s'' = r' + s + s''$. This is an equivalence relation and it can be seen that $[R] \ominus [S] := (R \times S)/\!\sim$ is an abelian semigroup in which every element of the form $[(s_1, s_2)]$ where $s_1, s_2 \in S$ is invertible with inverse $[(s_2, s_1)]$. We can then see that $\mathbb{Q} = [\mathbb{Z}] \ominus [\mathbb{N}]$ where the group operation $+$ is multiplication.

Why is $G(R)$ called a "universal" group? We shall see that it is a unique object in some sense for any abelian semigroup R. In fact, this is true even in the more general set up of $[R] \ominus [S]$, where S is a sub-semigroup of R.

Theorem 0.1. *Let H be a semigroup with identity and $\phi : R \to H$ be a homomorphism of semigroups that maps S into invertible elements of H. Then ϕ extends uniquely to a homomorphism $\psi : [R] \ominus [S] \to H$ such that the following diagram commutes:*



Proof. First we prove the uniqueness of ψ. Supose $\exists\, \psi_1, \psi_2 : [R] \ominus [S] \to H$ that both extend ϕ. Then

$$\begin{aligned}\psi_1([(r,s)]) &= \psi_1(([r] \ominus [s])) \\ &= \psi_1([r]) - \psi_1([s]) \\ &= \phi(r) - \phi(s) \\ &= \psi_2([(r,s)]).\end{aligned}$$

Define $\psi : [R] \ominus [S] \to H$ by

$$\psi([(r,s)]) := \phi(r) - \phi(s).$$

This is well-defined because $\phi(s)$ is invertible $\forall s \in S$, and if $(r,s) \sim (r', s')$, then $\exists\, s'' \in S$ such that

$$r + s' + s'' = r' + s + s''.$$

Hence

$$\phi(r) + \phi(s') + \phi(s'') = \phi(r') + \phi(s) + \phi(s'')$$

and $\phi(s'')$ can be cancelled from both sides. Further, ψ can be seen to be a homomorphism and $\psi(i(a)) = \psi([a]) = \psi([(a,0)]) = \phi(a) - \phi(0) = \phi(a)$. ☐

Corollary 0.2. *Let R_1, R_2 be abelian semigroups with identity elements. For any homomorphism $\phi : R_1 \to R_2$, $\exists$ a homomorphism $\psi : G(R_1) \to G(R_2)$ such that the following diagram commutes:*

$$\begin{array}{ccc} R_1 & \xrightarrow{\ \phi\ } & R_2 \\ \big\downarrow{\scriptstyle i_1} & & \big\downarrow{\scriptstyle i_2} \\ G(R_1) & \overset{\psi}{\dashrightarrow} & G(R_2) \end{array}$$

Proof. $\tilde{\phi} := i_2 \circ \phi : R_1 \to G(R_2)$ is a homomorphism such that the image of every element is invertible. Hence the previous theorem can be applied. ☐

The Grothendieck group construction is fundamental to a branch of mathematics known as **K-theory**. In particular, in order to study the K-theory of C^*-algebras, let A be a unital C^*-algebra. A C^*-algebra is a $*$-closed, norm closed subalgebra of $B(H)$ for a Hilbert space H. Let

$$E_\infty(A) := \sqcup_{n=1}^{\infty} \{e \in M_n(A) : e^2 = e\}.$$

Here, $M_n(A)$ denotes the set of $n \times n$ matrices with entries from A. We define a relation on $E_\infty(A)$ as:

$$e \sim f \iff \exists\, x, y \in A \text{ such that } xy = e, yx = f.$$

Then $\sim$ is an equivalence relation on $E_\infty(A)$. Let

$$V(A) := E_\infty(A)/\sim.$$

Then $V(A)$ is an abelian semigroup with identity, with addition defined as

$$[e] + [f] = [\begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix}]$$

$K_0(A)$ is then defined as the Grothendieck group of $V(A)$.

Another group called $K_1(A)$ can be defined using unitaries instead of idempotents. These groups are the beginnings of a vast area of mathematics, namely, operator K theory.

References

[1] N. E. Wegge-Olsen, *K-Theory and C^*-Algebras- A Friendly Approach*, Oxford University Press, 1993.