NOTES ON VON NEUMANN ALGEBRAS

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0.1. **Topologies on $B(H)$.** Let H be a complex separable Hilbert space and $B(H)$ be the $*$-algebra of bounded operators on H. $A \subseteq B(H)$ is a C^* algebra if and only if it is closed in the norm topology. We define some other topologies on $B(H)$.

Definition 0.1. We say that a net $\{x_\lambda\}_{\lambda\in\Lambda}$ in $B(H)$ converges strongly to $x \in B(H)$ if $\|x_\lambda\xi - x\xi\| \to 0\,\forall\xi \in H$. This is denoted by $x_\lambda \xrightarrow{st} x$.

Equivalently, we can define a strong neighbourhood as follows: For $\xi_1, \xi_2, \cdots, \xi_n \in H, \epsilon > 0, x \in B(H)$, let $N(x, \xi_1, \xi_2, \cdots, \xi_n, \epsilon) = \{y \in B(H) : \|x\xi_i - y\xi_i\| < \epsilon\,\forall i \in \{1, 2, \cdots, n\}\}$. Then this gives a basis for the strong topology, also called the strong operator topology or SOT.

Definition 0.2. We say that a net $\{x_\lambda\}_{\lambda\in\Lambda}$ in $B(H)$ converges weakly to $x \in B(H)$ if $\langle x_\lambda\xi, \eta\rangle \to \langle x\xi, \eta\rangle\,\forall\xi, \eta \in H$. This is denoted by $x_\lambda \xrightarrow{w} x$.

Equivalently, a weak neighbourhood is given as follows: For $\xi_1, \xi_2, \cdots, \xi_n \in H$ and $\eta_1, \eta_2, \cdots, \eta_n \in H, \epsilon > 0, x \in B(H)$, let $N(x, \xi_1, \xi_2, \cdots, \xi_n, \eta_1, \eta_2, \cdots, \eta_n, \epsilon) = \{y \in B(H) : |\langle(x - y)\xi_i, \eta_i\rangle| < \epsilon\,\forall i \in \{1, 2, \cdots, n\}\}$. This gives a basis for the weak or weak operator topology (WOT). Then the weak topology is contained in the strong topology, which is contained in the norm topology.

Exercise 0.3.
(1) Let $\{x_\lambda\}$ be a net such that $\sup\limits_{\lambda\in\Lambda} \|x_\lambda\| < \infty$. Let $S \subseteq H$ be total. Then $x_\lambda \xrightarrow{st} x$ iff $x_\lambda\xi \to x\xi\,\forall\xi \in S$. Similarly, $x_\lambda \xrightarrow{w} x$ iff $\langle x_\lambda\xi, \eta\rangle \to \langle x\xi, \eta\rangle\,\forall\xi, \eta \in S$.
(2) Let $H = l^2(\mathbb{N})$ and $S((x_1, x_2, \cdots)) = (0, x_1, x_2, \cdots)$. Let $S_n = S^n$. Then $S_n^* \xrightarrow{st} 0$ but $Sn \nrightarrow 0$ strongly.
    Similarly,for a continuous version of the above, let $H = L^2((0, \infty))$ and $S_t f(s) = \begin{cases} f(s-t), & s \geq t, \\ 0, & \text{otherwise.} \end{cases}$ Then $S_t^* \xrightarrow{st} 0$ but S_t does not converge to 0 strongly. Hence the $*$ operation is not strongly continuous.
(3) $x \to x^*$ is weakly continuous.
(4) If $x_\lambda \underset{w}{\xrightarrow{st}} x$, then $x_\lambda y \underset{w}{\xrightarrow{st}} xy$ and $yx_\lambda \underset{w}{\xrightarrow{st}} yx$. That is, multiplication is separately continuous.
(5) If $\{x_\lambda\}$ is a net such that $\sup\limits_{\lambda\in\Lambda} \|x_\lambda\| < \infty$ and $\{y_\lambda\}$ such that $\sup\limits_{\lambda\in\Lambda} \|y_\lambda\| < \infty$, and $x_\lambda \xrightarrow{st} x, y_\lambda \xrightarrow{st}$, then $x_\lambda y_\lambda \xrightarrow{st} xy$. So for bounded nets, multiplication is jointly continuous.
(6) If $S \subseteq B(H)$ is bounded, then the strong and topologies on S are metrizable. Choosing an orthonormal basis $\{\xi_n\}$ for H, we can write the metric as

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \|x\xi_n - y\xi_n\|$$

and

$$d(x, y) = \sum_{m,n=1}^{\infty} \frac{1}{2^{m+n}} |\langle(x - y)\xi_n, \xi_m\rangle|$$

for the strong and weak topologies respectively.

(7) Let $H_0 \subseteq H$ be a closed subspace and p be the projection onto H_0. Then TFAE for $x \in B(H)$:
   (a) $xH_0 \subseteq H_0$
   (b) $pxp = xp$

   Similarly, TFAE for $x \in B(H)$:
   (a) $xH_0 \subseteq H_0, x^*H_0 \subseteq H_0$
   (b) $px = xp$

(8) Let $A \subseteq B(H)$ be a $*$ algebra. Then TFAE:
   (a) $xH_0 \subseteq H_0 \,\forall x \in A$
   (b) $p \in A' = \{x \in B(H) : xx' = x'x \,\forall x \in A\}$

0.2. **Commutants.** Let S be a subset of $B(H)$. Then define the commutant of S as $S' = \{x \in B(H) : xx' = x'x \,\forall x \in S\}$. We define S'' as the commutant of S' and so on. Now, clearly, $S \subseteq S''$. This implies that $S''' \subseteq S'$. But substituting S by S' in the first inclusion, we get $S' \subseteq S'''$. Hence $\forall n \in \mathbb{N}$, S^{2n+1} are all equal, and S^{2n} are equal for $n \geq 1$. The question, then, is: When is $S = S''$? This is answered by the von Neumann density theorem. We first look at the finite dimensional version.

Proposition 0.4. *Let $A \subseteq B(H)$ be a unital $*$-subalgebra, where H is of finite dimension, say n. Then $A = A''$.*

Proof. Clearly $A \subseteq A''$. Let $y \in A''$. In order to show that $y \in A$, we will show that $\exists x \in A$ such that $x\xi_i = y\xi_i$ for any arbitrary n vectors in H. We embed A into $B(H \otimes H) = B(\oplus_{i=1}^n H_i)$ (where each $H_i = H$) by means of the map $\Pi : A \to B(H \otimes H)$ given by $\pi(x) = x \otimes 1$. Let $\xi = (\xi_1, \cdots, \xi_n) \in H \otimes H$. Let $K = \Pi(A)\xi = \{(x \otimes 1)\xi : x \in A\}$. Let $p_k : H \otimes H \to K$ be the projection onto K. As K is invariant under $\Pi(A)$, $p_k \in \Pi(A)'$. Hence $\Pi(A)''p_k = p_k\Pi(A)''$ and so K is invariant under $\Pi(A)''$. Now $\Pi(A) = \{\begin{bmatrix} x & 0 & \cdots & 0 \\ 0 & x & \cdots & 0 \\ \vdots & \cdots & & x \end{bmatrix} : x \in A\}$.

Exercise 0.5. Show that $\Pi(A)' = M_n(A')$ and $\Pi(A)'' = \Pi(A'')$.

Now, for $y \in A'', \eta \in K, \exists x \in A$ such that $\Pi(y)\eta = \Pi(x)\xi$ since K is invariant under $\Pi(A)''$. In particular, take $\eta = \xi$. Then $\exists x \in A$ such that $\Pi(y)\xi = \Pi(x)\xi$. But this implies that $y\xi_i = x\xi_i \,\forall i \in \{1, 2, \cdots, n\}$. □

Theorem 0.6 (von Neumann density theorem)**.** *Let $A \subseteq B(H)$ be a unital $*$-subalgebra. Then $\bar{A}^{st} = A''$, where $\bar{A}^{st}$ denotes the strong closure of A.*

Proof. Clearly, $A \subseteq A''$ and A'' is strongly closed. Let $y \in A''$ and let $\xi_1, \cdots, \xi_n \in H, \epsilon > 0$. We will show that $\exists x \in A$ such that $x \in N(y, \xi_1, \cdots, \xi_n, \epsilon)$. Embed A into $B(H \otimes \mathbb{C}^n) = B(\oplus_{i=1}^n H_i)$, where each $H_i = H$. Let Π denote the embedding. Then as earlier, $\Pi(A)' = M_n(A')$. For $\xi = (\xi_1, \cdots, \xi_n)$, let $K = \overline{\Pi(A)\xi}$. Then K is invariant under $\Pi(A)$ and hence under $\Pi(A)''$, as earlier. For $y \in A''$, $\Pi(y)\xi \in K$ and hence, for $\epsilon > 0$, $\exists x \in A$ such that $\|\Pi(x)\xi - \Pi(y)\xi]\| < \epsilon$. But this implies that $\|y\xi_i - x\xi_i\| < \epsilon \,\forall i \in \{1, \cdots, n\}$. □

Corollary 0.7 (von Neumann double commutant theorem)**.** *TFAE for a unital $*$-algebra.*

(1) $M = M''$
(2) M is strongly closed.
(3) M is weakly closed.

Definition 0.8. A unital $*$-subalgebra of $B(H)$ that satisfies one of the above equivalent conditions is called a von Neumann algebra.

Example 0.9.
(1) Any finite dimensional unital $*$-algebra is a vNa.
(2) $B(H)$ is a vNa.
(3) $L^\infty([0,1],\mathcal{B},\lambda)$ where λ is the Lebesgue measure on the Borel σ-algebra $\mathcal{B}$. For $f \in L^\infty$, the corresponding multiplication operator $M_f \in B(L^2([0,1))$. The claim is that $M = \{M_f : f \in L^\infty\}$ is a von Neumann algebra, that is $M = M''$. To prove this, it is sufficient to show that M is a maximal abelian subalgebra of $B(H)$, which means $M = M'$. Let $T \in M' \subseteq B(L^2)$. We want $f_0 \in L^\infty$ such that $M_{f_0} = T$. We define $f_0 := T1$. Since L^∞ is dense in L^2, it is sufficient to show that $Tf = M_{f_0} f \,\forall f \in L^\infty$. Now $Tf = TM_f 1 = M_f T1 = M_f f_0 = f_0 f$. Finally, we must show that $f_0 \in L^\infty$. We will prove that $\lambda(\{t \in [0,1] : |f_0(t)| > \|T\|\}) = 0$. This happens iff $\lambda(\{t \in [0,1] : |f_0(t)| \geq \|T\| + \frac{1}{n}\}) = \lambda(E_n) = 0 \,\forall n$. Suppose not, then $\lambda(E_n) > 0$ for some n. Let $\xi_n = \frac{1_{E_n}}{(\lambda(E_n))^{\frac{1}{2}}}$, a unit vector in L^∞. Then $\|T\xi_n\| \leq \|T\|$. But $\|T\xi_n\| = \|f_0\xi_n\| \geq \|T\| + \frac{1}{n}$, a contradiction.

Exercise 0.10. Show that $\{M_{\tilde{x}} : \tilde{x} \in l^\infty\} \subseteq B(l^2(\mathbb{N}))$ is a maximal abelian subalgebra.

0.3. **Group von Neumann algebras.**

Definition 0.11. A vNa M is called a factor if $Z(M) = M \cap M' = \mathbb{C}1$.

Example 0.12. $M_n(\mathbb{C})$ and $B(H)$ are examples of factors. We consider another example. Let G be a locally compact Hausdorff group. Then $\exists$ a unique (upto scalar) Haar measure on G, i.e. a measure μ on G such that

$$\int_G f(st)\, d\mu(t) = \int_G f(t)\, d\mu(t) \quad \forall f \in L^1(G,\mu).$$

Define for $g \in G, f \in L^2(G)$,

$$(u_g f)(g') = f(g^{-1}g'),\ g' \in G.$$

Exercise 0.13. Each u_g is a unitary with $u_g^* = u_{g^{-1}}$.

Definition 0.14. We define the group von Neumann algebra as

$$\lambda(G) = \{u_g : g \in G\}''.$$

From now on, we consider G to be a countable discrete group. Let $H = l^2(G)$. Let

$$e_g(g') = \delta_{g,g'} = \begin{cases} 1 & g = g' \\ 0 & \text{otherwise} \end{cases}$$

Then $\{e_g : g \in G\}$ is an orthonormal basis for $l^2(G)$. If u_g is defined as above, i,e, $(u_g f)(g') = f(g^{-1}g')$, then it is seen that $u_g e_{g'} = e_{gg'}$.

Example 0.15. Let $G = (\mathbb{Z}_n, +)$. Then $H = l^2(G) = \mathbb{C}^n$. Its orthonormal basis is given by

$$e_m(k) = \delta_{m,k} = \begin{cases} 1 & m = k \\ 0 & m \neq k \end{cases}$$

Then $u_k e_m = e_{(k+m) \text{mod } n}$. It can be seen that the u_ns will be matrices that are constant ($= 1$) on the diagonals. $\lambda(G) = \text{span}\,\{u_g : g \in G\}$ and hence if $n = 4$, say, $\lambda(G)$ consists of elements of the form

$$\begin{bmatrix} a & b & c & d \\ d & a & b & c \\ c & d & a & b \\ b & c & d & a \end{bmatrix}.$$

Since G is commutative, so is $\lambda(G)$.

Any $T \in B(l^2(G))$ can be considered as an infinite matrix indexed by $G \times G$. If $x \in \lambda(G)$, we denote the associated matrix also by x with $x(g, g') = \langle xe_{g'}, e_g \rangle$.

Proposition 0.16. *$x \in \lambda(G) \subseteq B(l^2(G))$ can be given in matrix form with respect to the orthonormal basis $\{e_g\}$. Then $x(g, g') = x(h, h')$ if $gg'^{-1} = hh'^{-1}$.*

Proof. Let $x = u_k$. $u_k(g, g') = \langle u_k e_{g'}, e_g \rangle = \langle e_{kg'}, e_g \rangle = \delta_{kg',g} = \delta_{k,gg'^{-1}}$. Hence if $gg'^{-1} = hh'^{-1}$, then $u_k(g, g') = u_k(h, h')$. The relation holds for finite linear combinations and extends to the weak closure $\lambda(G)$. □

Exercise 0.17. $M = \{x \in B(l^2(G)) : \exists c : G \to \mathbb{C}$ such that $x(g, g') = c(gg'^{-1})\}$ is a von Neumann algebra. (Hint: If c is the corresponding function for $x \in M$, choose the function given by $c^*(g) = \overline{c(g^{-1})}$ for x^* and convolve for products).

Proposition 0.18. *Let $M = \{x \in B(l^2(G)) : \exists c : G \to \mathbb{C}$ such that $x(g, g') = c(gg'^{-1})\}$. Then $\lambda(G) = M$.*

Proof. For $x \in \lambda(G)$, take $c = xe_1$, where 1 is the identity of G. This proves $\lambda(G) \subseteq M$.

For the reverse inclusion, let $x' \in \lambda(G)'$ and $x \in M$. We formally write $x = \sum_{h \in G} c_h u_h$. By this we (only) mean $xe_g = \sum_{h \in G} c_h e_{hg}$. Then we can directly verify (exercise!)

$$\langle x' x e_g, e_{g'} \rangle = \langle x x' e_g, e_{g'} \rangle.$$

Thus it follows that $\lambda(G)' \subseteq M'$. (This proof for the reverse inclusion was pointed out by Debdyuti in the tutorial.) □

Definition 0.19. Given $c \in l^2(G)$, consider $(c \star f)(g') = \sum_{g \in G} c(g) f(g^{-1} g')$, $f \in l^2(G)$. If $c \star f \in l^2(G)$, then define $L_c(f) = c \star f$.

Proposition 0.20. *L_c is bounded on $l^2(G)$.*

Proof. Let $f_n \in l^2(G)$ be such that $f_n \to 0$ and $c \star f_n \to f_0$. By the closed graph theorem, it suffices to prove that $f_0 = 0$. Now,

$$|(c \star f_n)(g')| \leq |\sum_{g \in G} c(g) f_n(g^{-1} g')| \leq \|c\|_2 \|f_n\|_2 \, \forall g' \in G,$$

by Cauchy Schwarz. Hence $\|c \star f_n\|_\infty \leq \|c\|_2 \|f_n\|_2$. As $f_n \to 0$, $\|f_n\|_2 \to 0$. Hence $c \star f_n \to 0$ in l^∞. Hence $c \star f_n \to 0$ in l^2, since it is already known to converge to some f_0 in l^2. □

Definition 0.21. For c as above, let $\lambda C(G) = \{L_c : c \star f \in l^2(G) \, \forall f \in l^2(G)\}''$.

We have shown that $\lambda(G) = M \subseteq \lambda C(G)$. Indeed it is an equality which will be clear after proving Tomita-Takesaki theorem (for II_1 factors).

Definition 0.22. A group G is said to be ICC (infinite conjugacy class) if all the conjugacy classes are infinite except for the identity.

Proposition 0.23. *$\lambda(G)$ is a factor iff G is ICC.*

Proof. Suppose G is ICC. Let $x \in \lambda(G) \cap \lambda(G)'$. We must show that x is a scalar. Since $\lambda(G) = M$, we write x as the (formal) sum $x = \sum_{g' \in G} c_{g'} u_{g'}$.

Now, $xu_h = u_h x \, \forall h \in G$. Thus $u_h x u_h^* = x$. Hence $\langle u_h x u_h^* e_1, e_g \rangle = \langle x u_h^* e_1, u_h^* e_g \rangle = \langle x e_{h^{-1}}, e_{h^{-1}g} \rangle = \langle \sum_{g' \in G} c_{g'} u_{g'} e_{h^{-1}}, e_{h^{-1}g} \rangle = c_{h^{-1}gh}$.

On the other hand, $\langle xe_1, e_g \rangle = c_g$. Hence, for each $g \in G$, $c_{h^{-1}gh} = c_g \, \forall h \in G$. So c is constant on conjugacy classes. But since $c = xe_1 \in l^2(G)$, we must have $c = 0$.

Conversely, if G is not ICC. Let C be a finite conjugacy class. Define $x = \sum_{g \in C} u_g$. Now, $\sum_{g \in C} u_{h^{-1}gh} = \sum_{g \in C} u_g \, \forall h \in G$, since C is a conjugacy class.

This implies that $u_h x u_h^* = x \,\forall h \in G$. Hence $xu_h = u_h x \,\forall h \in G$, so x is a non scalar which is in $\lambda(G) \cap \lambda(G)'$.

□

Definition 0.24. We define the trace on $\lambda(G)$ by $\mathrm{tr} : \lambda(G) \to \mathbb{C}$, $\mathrm{tr}(x) = \langle xe_1, e_1 \rangle$.

Proposition 0.25. *The functional tr is linear, weakly continuous, tracial (i.e. tr(xy)=tr(yx)), positive (*$tr(x^*x) \geq 0 \,\forall x \in \lambda(G)$*), and faithful (*$tr(x^*x) = 0 \Rightarrow x = 0$*). Also,* $tr(1) = 1$.

Proof. It is easily seen that tr is linear and weakly continuous. We show that it is tracial as follows: If $x = \sum_{g\in G} c_g u_g$ and $y = \sum_{g\in G} d_g u_g$, then

$$xy = \sum_{h\in G} (\sum_{g\in G} c_g d_{g^{-1}h}) u_h.$$

Hence, $\mathrm{tr}(xy) = \langle xye_1, e_1 \rangle = \sum_{g\in G} c_g d_{g^{-1}h} = \sum_{g\in G} d_g c_{g^{-1}h} = \mathrm{tr}(yx)$.

To prove positivity, suppose $\mathrm{tr}(x^*x) = 0$. Then $\sum_{g\in G} |c_g|^2 = 0 \Rightarrow c_g = 0 \,\forall g \in G \Rightarrow x = 0$. Finally, $\mathrm{tr}(1) = \langle e_1, e_1 \rangle = 1$. □

Exercise 0.26.

(1) On $M_n(\mathbb{C})$, define tr: $M_n(\mathbb{C}) \to \mathbb{C}$ by
$$\mathrm{tr}((x_{ij})) = \frac{1}{n} \sum_{i=1}^{n} x_{ii}.$$
Then tr is the unique linear functional on $M_n(\mathbb{C})$ satifying $\mathrm{tr}(xy) = \mathrm{tr}(yx)$, $\mathrm{tr}(1) = 1$.

(2) Let H be an infinite dimensional Hilbert space. Then show that there exists no weakly continuous linear functional tr: $B(H) \to \mathbb{C}$ satisfying $\mathrm{tr}(xy) = \mathrm{tr}(yx) \,\forall x, y \in H$.

Exercise 0.27. If the the trace on a von Neumann algebra is unique, then it is a factor. (If there exists a nontrivial central projection p consider $\frac{t}{tr(p)} tr(xp) + \frac{1-t}{tr(1-p)} tr(x(1-p))$ for $t \in (0, 1)$.

Definition 0.28. Let M, N be von Neumann algebras. An isomorphism $\Phi : M \to N$ is a bijective linear $*$-homomorphism which is weakly continuous. (Actually the weakly continuous condition can be dropped from this definition.)

Proposition 0.29. *If G is ICC, then* $\lambda(G)$ *is an infinite dimensional factor not isomorphic to* $B(H)$ *for any* H.

Proof. $\lambda(G)$ is infinite dimensional since the set $\{u_g : g \in G\}$ is linearly independent. The proposition follows since there exists a trace on $\lambda(G)$. □

Proposition 0.30. $\lambda(G)$ *does not contain isometries which are not unitaries.*

Proof. Suppose $u \in \lambda(G)$ and $u^*u = 1$. Then
$$0 \leq \mathrm{tr}(1 - uu^*) = 1 - \mathrm{tr}(uu^*) = 1 - \mathrm{tr}(u^*u) = 0.$$
Since u is an isometry, $1 - uu^*$ is a projection and hence positive. Thus $\mathrm{tr}(1 - u^*u) = 0 \Rightarrow 1 - uu^* = 0$. This is called the finiteness property of von Neumann algebras. □

We list some facts we will require.

(1) Projections generate any von Neumann algebra. This follows from spectral theorem.
(2) Let A be a C^* algebra. Then any $x \in A$ can be written as a linear combination of four unitaries. In particular, if x is self adjoint and $\|x\| \leq 1$, define $u = x + i\sqrt{1 - x^2}$. Then $x = \frac{u+u^*}{2}$.

(3) For a von Neumann algebra M, let $P(M)$ denote the set of all projections in M and $U(M)$ denote the set of unitaries in M. Then$M = (P(M'))' = U((M'))'$.

Definition 0.31. Let $x \in B(H)$. The left support of X is defined as the projection onto $\overline{\text{range}(x)}$ and is denoted by $l(x)$. The right support of X is defined as the projection onto $(\ker(x))^{\perp}$ and is denoted by $r(x)$. If the two are equal, they are called the support of x.

Exercise 0.32. $l(x)$ is the smallest projection p such that $px = x$, and $r(x)$, the smallest projection such that $xp = x$.

Proposition 0.33. *If $x \in M$, then $l(x), r(x) \in M$.*

Proof. Suppose a projection satisfies $px = x$. Then for any $u' \in U(M')$, $u'pxu'^* = u'xu'^* = u'u'^*x = x$. Hence $px = x$ iff $u'pxu'^* = x \, \forall u' \in U(M')$. In particular, $u'l(x)u'^*x = x \, \forall u' \in U(M')$. Hence, by the exercise, $l(x) \leq u'l(x)u'^* \, \forall u' \in U(M')$. But this implies that $l(x)u' = u'l(x) \, \forall u' \in U(M')$. Hence $l(x) \in M$. Similarly, $r(x) \in M$. □

Exercise 0.34. Let $x \in M$ and $x = u(x^*x)^{\frac{1}{2}} = u|x|$ be the polar decomposition of x. Then $|x|, u \in M$.

Remark 0.35. Let p, q be projections. Then $p \wedge q$ is the projection onto $pH \cap qH$, and $p \vee q = (p^{\perp} \wedge q^{\perp})^{\perp}$ is the projection onto the space generated by pH and qH. If $p, q \in M$, then $p \wedge q, p \vee q \in M$.

Proposition 0.36. *Let $\{x_\lambda\}_{\lambda \in \Lambda} \subseteq M$ be an increasing net of self-adjoint elements, i.e. $x_{\lambda_1} \leq x_{\lambda_2}$ if $\lambda_1 \leq \lambda_2$. Let $\sup_{\lambda \in \Lambda} \|x_\lambda\| < \infty$. Then $\exists x \in M$ such that $x_\lambda \to x$ strongly.*

Proof. For each $\xi \in H$, $\langle x_\lambda \xi, \xi \rangle$ increases to some scalar. Let the (self-adjoint) operator determined by this quadratic form be called x. Then $\langle (x - x_\lambda)\xi, \xi \rangle \to 0$. Hence $\|(x - x_\lambda)^{\frac{1}{2}}\xi\| \to 0$. This implies that $(x - x_\lambda)^{\frac{1}{2}} \xrightarrow{\text{st}} 0$, which in turn implies that $(x - x_\lambda) \xrightarrow{\text{st}} 0$ because multiplication is jointly continuous strongly on uniformly bounded sets. □

Example 0.37. Some examples of ICC groups are:

(1) $S_\infty = \cup S_n = \{$permutations on $\{1, 2, \cdots\}$ fixing all but finitely many$\}$.
(2) $\mathbb{F}_n$, the free group generated by n elements.

Remark 0.38. Let $\{p_\lambda : \lambda \in \Lambda\} \subseteq M$ be a collection of projections. Let F denote all finite subsets of Λ. For $s \in F$, define $p_s = \vee_{\lambda \in S} p_\lambda$. Then $\{p_s\}_{s \in F}$ is an increasing net. Its limit is denoted by $\vee_{\lambda \in \Lambda} p_\lambda$, the projection onto the closed subspace generated by the ranges of all p_λ. Similarly, one can talk about $\wedge_{\lambda \in \Lambda} p_\lambda$, the projection onto the intersection of ranges of p_λ.

0.4. **Equivalence of projections.**

Proposition 0.39. *Let M be a von Neumann algebra in $B(H)$, and $p \in P(M)$. Then $pMp = \{pxp : x \in M\}$ and $M'p = \{px' : x' \in M'\}$ are von Neumann algebras in $B(pH)$.*

Proof. On pH, pMp and $M'p$ commute, as $(pxp)(x'p) = pxx'pp = px'xp = (x'p)(pxp)$. Hence, $pMp \subseteq (M'p)'$ and $M'p \subseteq (pMp)'$ on pH. We show that in both cases, equality holds, hence pMp and $M'p$ are von Neumann algebras.

Let $x \in (M'p)' \subseteq B(pH)$. We want $\tilde{x} \in M$ such that $p\tilde{x}p = x$. Let $\tilde{x} = xp$ on H. Then $\tilde{x} = xp = pxp$ on pH. Let $x' \in M'$. Then

$$\begin{aligned} x'\tilde{x} &= x'xp \\ &= x'pxp \\ &= (x'p)(pxp) \\ &= (pxp)(x'p) \\ &= \tilde{x}x' \text{ (on } pH). \end{aligned}$$

Hence $\tilde{x} \in M'' = M$. Next, we prove that $(pMp)' \subseteq M'p$. It is enough to prove this for the unitaries of $(pMp)'$. Let $u \in U(pMp)'$. Let $K = \overline{MpH} \subseteq H$, and $q : H \to K$ be the projection onto K. Then q commutes with both M and M', hence $q \in Z(M)$. We want $\tilde{u} \in M'$ such that $\tilde{u}p = u$. Define u^0 on K by

$$u^0(\sum m_i \xi_i) = \sum m_i u \xi_i,\ m_i \in M, \xi \in pH.$$

Then

$$\begin{aligned} \langle u^0(\sum_i x_i\xi_i), u^0(\sum_j y_j\eta_j)\rangle &= \langle \sum_i x_i u\xi_i, \sum_j y_j u\eta_j\rangle \\ &= \langle \sum_i x_i pu\xi_i, \sum_j y_j pu\eta_j\rangle \\ &= \sum_{i,j} \langle p y_j^* x_i pu\xi_i, u\eta_j\rangle \\ &= \sum_{i,j} \langle up y_j^* x_i p\xi_i, u\eta_j\rangle \\ &= \langle \sum_i x_i\xi_i, \sum_j y_j\eta_j\rangle. \end{aligned}$$

Hence, by the totality of the set $\{m_i\xi_i : m_i \in M, \xi \in pH\}$ in qH, u^0 is well-defined and extends to an isometry on qH. Now define $\tilde{u} = u^0 q$, and notice that u^0 commutes with M on qH. That is, $u^0 m\xi = m u^0 \xi\, \forall \xi \in qH$. Hence $\tilde{u} = u^0 q \in M'$ and $u = \tilde{u}p$. □

Corollary 0.40. *If M is a factor and $p \in M$, then pMp and $M'p$ are factors, and $x' \mapsto x'p$ is a weakly continuous $*$-isomorphism between M' and $M'p$.*

Proof. As in the previous proof, let $K = \overline{MpH} \subseteq H$, and $q : H \to K$ be the projection onto K. Then q commutes with both M and M', hence $q \in Z(M)$. As M is a factor, q must be identity, and hence $K = H$. Suppose, for $x' \in M'$, $x'p = 0$. Then $x'(\sum m_i\xi_i) = \sum m_i x' p \xi_i = 0$, $m_i \in M, \xi \in pH$. Hence $x' = 0$ on $K = H$. Thus the above map is injective, and $M'p$ is a factor. But $pMp = (M'p)'$ on pH, and hence is also a factor. □

Proposition 0.41. *Let M ba a factor, and $p, q \in P(M)$ be non-zero projections. Then $\exists u \in M$ unitary such that $puq \neq 0$.*

Proof. Suppose $\forall u \in U(M)$, $puq = 0$. Then $(u^*pu)q = 0\, \forall u \in U(M)$, where u^*pu is a projection. Hence $(\vee_{u\in U(M)} u^*pu)q = 0$. But for non-zero $p, \vee_{u \in U(M)} u^*pu \in M'$ and hence equals to 1. □

Proposition 0.42. *Let $p, q \in P(M)$ be non-zero projections. Then $\exists$ a partial isometry $u \neq 0$ such that $u^*u \leq q$ and $uu^* \leq p$.*

Proof. By the previous proposition, $\exists x$ such that $pxq \neq 0$. Let u be the partial isometry in the polar decomposition pf pxq. Then it can be verified (exercise!) that $u^*u \leq q$ and $uu^* \leq p$. □

Definition 0.43. Let M be a von Neumann algebra. $p \in P(M)$, a non-zero projection, is said to be minimal if whenever $0 \neq q \leq p$, for $q \in P(M)$, then $q = p$.

Corollary 0.44. *If $u \in M$ is a partial isometry such that u^*u is minimal, then uu^* is also minimal.*

Corollary 0.45. *If p is a minimal projection, then $pMp = \mathbb{C}p$.*

Definition 0.46. A factor M is said to be of type I if it contains minimal projections. $B(H)$ is a type I factor.

We prove that $B(H)$ is essentially the only kind of type I factor

Proposition 0.47. *If $M \subseteq B(H)$ is a type I factor, then $\exists H_0$ and K Hilbert spaces, and $U : H_0 \otimes K \to H$ unitary such that $U(B(H_0) \otimes 1)U^* = M$.*

Proof. Let e_1 be any minimal projection in M. Let $\{e_1, e_2, \cdots\}$ be a maximal collection of mutually orthogonal minimal projections in M. Claim: $\sum_n e_n = 1$. If not, choose a partial isometry u such that $u^*u \leq 1 - \sum_n e_n$ and $uu^* = e_1$ (possible by Proposition 0.42). $uu^* = e_1$ is minimal, and so u^*u is minimal, and $u^*u \perp e_n \,\forall n$, which contradicts the maximality of $\{e_1, e_2, \cdots\}$.

Using minimality, now choose partial isometry e_{1i} satisfying $e_{1i}e_{1i}^* = e_1$ and $e_{1i}^*e_{1i} = e_i$. Let $x \in M$, then $x = \sum e_i x e_j$, with the sum denoting strong convergence. Notice

$$e_i x e_j = e_{1i}^* e_{1i} x e_{1j}^* e_{1j} \in e_1 M e_1 = \mathbb{C}e_1.$$

Now the proposition follows from the following exercise.

Exercise 0.48. $H = \oplus_j e_j H$. Let $H_0 = e_1 H$ and $K = l^2(\mathbb{N})$. Define $U : H_0 \otimes l^2(\mathbb{N}) \to H$ by $U(\xi_0 \otimes f) = \oplus_j f(j) e_{j1} \xi_0$. Verify that U is a well-defined unitary that satisfies $U(B(H_0) \otimes 1)U^* = M$.

□

Corollary 0.49. *If G is ICC, then $\lambda(G)$ is not type I.*

Definition 0.50. Let M be a von Neumann algebra and $p, q \in P(M)$ be non-zero projections. Then $p \sim q$ if $\exists$ a partial isometry $u \in M$ such that $u^*u = q$ and $uu^* = p$. p and q are said to be Murray-von Neumann equivalent.

Exercise 0.51. $\sim$ is an equivalence relation.

Definition 0.52. An order is defined on $P(M)$ as $p \preceq q$ if $\exists$ a partial isometry u such that $uu^* = p$ and $u^*u \leq q$.

Proposition 0.53. *$\preceq$ is a partial order on $P(M)$, meaning that if $p \preceq q$ and $q \preceq p$, then $p \sim q$.*

Proof. Let $p \preceq q$ and $q \preceq p$. Then $\exists u, v$ such that $u^*u \leq q, uu^* = p$, and $v^*v \leq p, vv^* = q$. Define

$$\begin{array}{ll} p_0 = p & q_0 = q \\ p_1 = v^* q_0 v & q_1 = u^* p_0 u \\ \vdots & \vdots \\ p_{n+1} = v^* q_n v & q_{n+1} = u^* p_n u \end{array}$$

Note that $p_i \perp p_j$ and $q_i \perp q_j \,\forall i \neq j$. Claim: q_n and p_n are decreasing projections. The proof is by induction. Note that $p_1 = v^* q_0 v = v^* q v = v^* v v^* v \leq p = p_0$, and similarly $q_1 \leq q_0$. Next, suppose $p_{n-1} \geq p_n, q_{n-1} \geq q_n$. Then $v^* q_{n-1} v \geq v^* q_n v \Rightarrow p_n \geq p_{n+1}$. Similarly, $q_n \geq q_{n+1}$. Define $p_\infty = \wedge_{n=0}^{\infty} p_n$ and

$q_\infty = \wedge_{n=0}^{\infty} q_n$. For each i, let $w = p_i - p_{i+1}$ Then $u^*w^*wu = q_{i+1} - q_{i+2}$ and $wuu^*w^* = p_i - p_{i+1}$. Hence $q_{i+1} - q_{i+2} \sim p_i - p_{i+1}$. Then $u^*p_\infty p_\infty u = q_\infty$ and $p_\infty uu^*p_\infty = p_\infty$. Hence $p_\infty \sim q_\infty \Rightarrow p \sim q$. □

When M has a faithful weak continuous positive trace, here is an alternate proof.

Proof. $\mathrm{tr}(v^*v) \leq \mathrm{tr}(p) = \mathrm{tr}(uu^*) = \mathrm{tr}(u^*u) \leq \mathrm{tr}(q) = \mathrm{tr}(vv^*) = \mathrm{tr}(v^*v)$. Hence $\mathrm{tr}(p - v^*v) = 0$. Hence $p = v^*v \Rightarrow p \sim q$. □

Definition 0.54.
(1) $p \in P(M)$ is said to be infinite if $\exists q \lneq p$ such that $q \sim p$.
(2) $p \in P(M)$ is said to be finite if it is not infinite.
(3) A von Neumann algebra is said to be infinite if 1 is an infinite projection.
(4) A factor M is said to be type I if it contains non-zero minimal projections (eg. $B(H)$).
(5) A factor M is said to be type II if contains no minimal projections, but non-zero finite projections exist (eg. $\lambda(G)$ when G is ICC).
(6) A factor M is said to be type II_1 if it is type II and 1 is finite.
(7) A factor M is said to be type II_∞ if it is type II and 1 is infinite.
(8) A factor M is said to be type III if it has no finite projections.

Exercise 0.55. If 1 is finite then all projections are finite.

Proposition 0.56. *If M is a factor then $\preceq$ is a total order on $P(M)$.*

Proof. Let $p, q \in P(M)$. We must show that either $p \preceq q$ or $q \preceq p$. By Proposition 0.42, $\exists u$ partial isometry such that $uu^* \leq p, u^*u \leq q$.
Let $S = \{$partial isometries u such that $uu^* \leq p, u^*u \leq q\}$. Define a partial order on S by $u \leq v$ if $u^*u \leq v^*v$ and $v \restriction_{u^*uH} = u$. Under this partial order, every chain has an upper bound. Let u_0 be a maximal element. Claim: Either $u_0u_0^* = p$ or $u_0^*u_0 = q$. If not, $p - u_0u_0^* > 0$ and $q - u_0^*u_0 > 0$. Then, by Proposition 0.42, $\exists$ a partial isometry v such that $vv^* \leq p - u_0u_0^*, v^*v \leq q - u_0^*u_0$. Then $(u_0 + v)(u_0 + v)^* \leq p$ and $(u_0 + v)^*(u_0 + v) \leq q$, contradicting the maximality of u_0. □

Definition 0.57. A trace is said to be normal if $0 \leq x_\alpha \nearrow x \Rightarrow \mathrm{tr}(x_\alpha) \nearrow \mathrm{tr}(x)$.

0.5. Type II_1 factors.

Theorem 0.58. *Any II_1 factor admits a faithful normal positive trace. The converse also holds.*

Proposition 0.59. *Let M be a II_1 factor. Then given $\epsilon > 0, \exists$ a non-zero projection $p \in P(M)$ such that $tr(p) < \epsilon$.*

Proof. Suppose the proposition is false. Let $0 < d = \inf\{\mathrm{tr}(p) : p \in P(M), p \neq 0\}$. Let $\epsilon > 0$ be arbitrary. $\exists p \in P(M)$ such that $d \leq \mathrm{tr}(p) < d + \epsilon$. We know that M contains no minimal projection. Hence $\exists q \lneq p$ so that $p - q \neq 0$. Then $d \leq \mathrm{tr}(q)$. But $\mathrm{tr}(p - q) < d + \epsilon - \mathrm{tr}(q) \leq \epsilon$. But this is a contradiction as ϵ is arbitrary and can be chosen smaller than d. □

Lemma 0.60. *Let M be type II_1 and $0 \neq p \in P(M)$. Then pMp is also a type II_1 factor.*

Proof. Define $\mathrm{tr}_p(pmp) = \frac{\mathrm{tr}(pmp)}{\mathrm{tr}(p)}$, $m \in M$. Then tr_p is a trace on pMp, hence pMp is II_1. □

Proposition 0.61. *Let M be type II_1. Then tr: $P(M) \to [0, 1]$ is onto.*

Proof. Let $r \in [0,1]$. We want $p \in P(M)$ such that $\mathrm{tr}(p) = r$. Let $S = \{p \in P(M) : \mathrm{tr}(p) \leq r\}$ with the usual order for self-adjoints. By Zorn's lemma, S has a maximal elements, say q. Claim: $\mathrm{tr}(q) = r$. Suppose not. Then $(1-q)M(1-q)$ is type II by Lemma 0.60. By Proposition 0.59, $\exists$ a projection $p \in (1-q)M(1-q)$ such that $\mathrm{tr}(p) < r - \mathrm{tr}(q)$. Then $p + q \geq q$ and $\mathrm{tr}(p+q) \leq r$, contradicting the maximality of q. □

Corollary 0.62. *Let M be a type II_1 factor. Then $tr : P(M)/\sim \to [0,1]$ is an order isomorphism, i.e., it is $1-1$, onto and order preserving.*

Proof. The map is $1-1$ as $\preceq$ is a total order on $P(M)/\sim$. □

Definition 0.63.
(1) $\omega \in H$ is said to be cyclic for M if $[M\omega] = H$, where $[M\omega] = \overline{\{x\omega : x \in M\}}$.
(2) $\omega \in H$ is said to be separating for M if $x \in M, x\omega \Rightarrow x = 0$.

Proposition 0.64. *Let $M \subseteq B(H)$ be a von Neumann algebra. Then $\omega \in H$ is cyclic for M iff ω is separating for M'.*

Proof. Let ω be cyclic for M and $x'\omega = 0$ for $x' \in M'$. Then $x'x\omega = xx'\omega = 0\,\forall x \in M$. Hence $x' = 0$. Next, suppose ω is separating for M'. Let $H_0 = [M\omega]$. Let $p : H \to H_0$. Since p leaves H_0 invariant, $p \in M'$, and so $1 - p \in M'$. Then $(1-p)\omega = 0 \Rightarrow 1 - p = 0$. Hence $H_0 = H$. □

Corollary 0.65. *Let M be a von Neumann algebra. Then ω is cyclic and separating for M iff it is cyclic and separating for M'.*

Definition 0.66. Let $M \subseteq B(H)$ be a II_1 factor with a trace tr. We say that M is in standard form if $\exists \omega \in H$ which is cyclic and separating for M such that $\mathrm{tr}(x) = \langle x\omega, \omega\rangle$.

Example 0.67.
(1) $\Lambda(G) \subseteq B(l^2(G))$ is in standard form with $\omega = e_1$. Let $x = \sum c_g e_g \in \lambda(G)$. $xe_1 = 0 \Rightarrow \sum c_g e_g = 0$. Hence $x = 0$. This shows that e_1 is separating. e_1 can be shown to be cyclic by considering finitely supported $l^2(G)$ functions.
(2) Let M be a II_1 factor with a trace tr. Define $\langle x, y\rangle = \mathrm{tr}(y^*x)$ and let $H = l^2(M)$ be the closure of M with respect to this inner product. Let $\omega = 1 \in \bar{M}$ and $\Pi_{\mathrm{tr}} : M \to B(H)$ be the associated representation. Then $\Pi_{\mathrm{tr}}(M)$ is in standard form.

From now, we assume that $M \subseteq B(H)$ is a type II_1 factor in standard form. Define $J_M(x\omega) = x^*\omega$, $x \in M$. Then J_M is antilinear and $J_M^2 = 1$.

Proposition 0.68. *J_M extends to H as an anti-unitary.*

Proof. $\langle J_M x\omega, J_M y\omega\rangle = \langle x^*\omega, y^*\omega\rangle = \mathrm{tr}(yx^*) = \mathrm{tr}(x^*y) = \langle y\omega, x\omega\rangle$. □

Proposition 0.69. *$J_M x^* J_M(y\omega) = yx\omega$, $x, y \in M$.*

Hence the operator $J_M x^* J_M$ acts as right multiplication by x.

Proposition 0.70. *$J_M M J_M \subseteq M'$.*

The proof follows since elements of M act as left multiplication, $J_M x^* J_M$ acts as right multiplication and by associativity. Indeed, equality holds in the above proposition (Tomita-Takesaki theorem).

Proposition 0.71. *Let M be in standard form with $\omega \in H$ cyclic and separating. Then $J_M(x'\omega) = x'^*\omega$ and $x' \mapsto \langle x'\omega, \omega\rangle$ is a trace on M'. Hence M' is also in standard form and $J_M = J_{M'}$.*

Proof. Let $x \in M, x' \in M'$. Then

$$\begin{aligned}\langle J_M(x'\omega), x\omega\rangle &= \langle J_M x\omega, J_M J_M(x'\omega)\rangle \\ &= \langle J_M x\omega, x'\omega\rangle \\ &= \langle x^*\omega, x'\omega\rangle \\ &= \langle \omega, xx'\omega\rangle \\ &= \langle \omega, x'x\omega\rangle \\ &= \langle x'^*\omega, x\omega\rangle.\end{aligned}$$

Hence $J_M(x') = x'^*\omega$ and $J_M = J_{M'}$. Further,

$$\begin{aligned}\langle x'y'\omega, \omega\rangle &= \langle y'\omega, x'^*\omega\rangle \\ &= \langle Jx'^*\omega, Jy'\omega\rangle \\ &= \langle x'\omega, y'^*\omega\rangle \\ &= \langle y'x'\omega, \omega\rangle.\end{aligned}$$

Hence the tracial property is satisfied. □

Corollary 0.72. *If M is a II_1 factor in standard form, then $M' \subseteq B(H)$ is a II_1 factor in standard form, and $JMJ = M'$, i.e., M' is anti-isomorphic to M*

Proof. Let ϕ be the map that sends x to Jx^*J. Then $\phi(xy) = \phi(y)\phi(x)$. □

We can also associate a von Neumann algebra to a group through right multiplication as follows. For a countable group G, define $v_h e_g = e_{gh-1}$, and extend to $l^2(G)$. Let $\rho(G) = \{v_g : g \in G\}''$; we can also define $\rho C(G)$ by right convolution operators similar to $\lambda C(G)$. Then $\rho(G) \subseteq \rho C(G)$ and hence $\rho C(G)' \subseteq \rho(G)'$.

We have proved $\lambda(G) \subseteq \rho(G)'$; also $\lambda C(G) \subseteq \rho C(G)'$. Now it is easy to verify directly using Tomita-Takesaki theory for II_1 factors that $\lambda(G)' = \rho(G)$. Hence it follows that $\lambda(G) = \lambda C(G) = \rho C(G)' = \rho(G)'$.

We do not prove the following converse statement, but will be used.

Proposition 0.73. *Let M be a finite von Neumann algebra. Then there exists a faithful normal positive trace on M.*

0.6. Type II_∞ factors.

Exercise 0.74. Any projection $p \in M$, p is finite if and only if pMp is finite.

Lemma 0.75. *Let p, q are finite projections satisfying $p \perp q$. Then $p+q$ is also finite.*

Proof. Since both pMp and qMq are finite, there exists traces on them. Now we can define a trace on $(p+q)M(p+q)$ by $1/2(tr(pxp) + tr(qxq))$. □

Lemma 0.76. *Let M be type II_∞ and $p \in P(M)$ be finite. Then $\exists$ countably infinite projections $p_1, p_2, \cdots$ such that $p_n \sim p \,\forall n$ and $\sum_{n=1}^\infty p_n = 1$.*

Proof. Let $S = \{p_1, p_2, \cdots\}$ be the maximal family of orthogonal projections such that $p_n \sim p$ and $p_1 = p$. Then S has to be infinite since each p_n is a finite projection. Let $q = 1 - \sum_{n=1}^\infty p_n$. Then $q \preceq p$, for otherwise, maximality of S is contradicted. Now, $\sum_{n=1}^\infty p_n \sim \sum_{n=2}^\infty p_n$ (exercise!). Hence $1 = q + \sum_{n=1}^\infty p_n \preceq p_1 + \sum_{n=2}^\infty p_n \preceq 1$. Hence $\sum_{n=1}^\infty p_n \sim 1$. If u is the partial isometry (indeed an isometry) implementing this equivalence, then by replacing p_n by u^*p_n we may assume $\sum_{n=1}^\infty p_n = 1$. □

Proposition 0.77. *Let M be type II_∞. Then for any finite projection $p \in M$, pMp is type $II_1 \subseteq B(pH)$ and $pMp \otimes B(l^2(\mathbb{N})$ is isomorphic to M.*

Proof. pMp is a II_1 factor. By Lemma 0.76, $\exists$ countably infinite projections $p_1, p_2, \cdots$ such that $p_n \sim p\, \forall n$ and $\sum_{n=1}^{\infty} p_n = 1$. Then $\exists u_{i1}$ such that $u_{i1}^* u_{i1} = p_1$ and $u_{i1} u_{i1}^* = p_i$. Let $u_{ij} = u_{i1}^* u_{1j}$. The required map from M to $pMp \otimes B(l^2(\mathbb{N})$ can be defined by $m \mapsto ((u_{i1} m u_{j1}^*))$. $\square$

0.7. **Type III factors.** The following is a way to construct type III. Let $M_{2n} = M_{2n}(\mathbb{C})$. Consider the tower of algebras

$$A_1 = M_2 \hookrightarrow A_2 = M_4 \hookrightarrow \cdots \hookrightarrow A_n = M_{2^n} \hookrightarrow M_{2^{n+1}\cdots}$$

via the maps

$$x \mapsto x \otimes 1 \mapsto x \otimes 1 \otimes 1 \cdots .$$

Then we can consider $A^0 = \bigcup_{n=1}^{\infty} A_n$ as formal infinite tensors with 1 in all but finitely many places. A trace and norm is well-defined on this 'inductive limit', since that is preserved under the inclusion maps. Let π_{tr} be the associated GNS representation then $M = \pi_{tr}(A^0)''$ is a II_1 factor. (Reason: The vacuum state given by the GNS construction is indeed a trace, since its restriction to $\pi_{tr}(A^0)$ is a trace. It is a factor since this trace is unique, as its restriction is the unique trace on $\pi_{tr}(A^0)$.)

Now, instead of the trace if take our original state on M_2 as

$$\phi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \frac{1}{1+\lambda} a + \frac{\lambda}{1+\lambda} d,$$

for $\lambda \in (0, 1)$, then the same construction leads to type III factors.